Bifurcations

in the presence of noise

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Motivations to consider "noise" in dynamics

- System is driven by a signal with probabilistic characterisation.
- High-dimensional system admits a reduction to low-dimensional systems driven by noise (either theoretically justified or phenomenologically).
- Effective accounting for modelling uncertainty, considering a random ensemble of models to describe a system rather than one (arbitrary) model among them.

How are bifurcations affected by noise?

Consider a *stochastic differential equation (SDE)* on a space X

$$dx = f_\alpha(x)dt + \sigma dW_t\,,$$

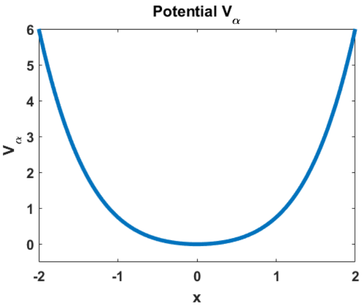
where $\alpha = 0$ is a bifurcation point for the deterministic system

$$\frac{dx}{dt} = f_\alpha(x).$$

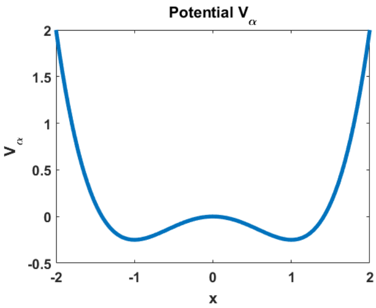
Question: Does the stochastic system exhibit a bifurcation? If so, in what sense?

Example: pitchfork bifurcation with additive noise

$f_\alpha(x) = -\partial_x V_\alpha(x)$ with $V_\alpha = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$.



: $\alpha = -1$



: $\alpha = 1$

From SDE to Random Dynamical System

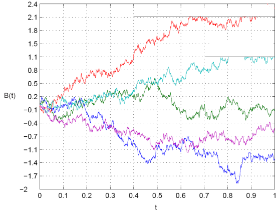
The SDE driven by additive noise

$$dx_t = f(x_t)dt + \sigma\, dW_t,$$

can be viewed as a random dynamical system (skew-product flow) ϕ satisfying

$$\phi(t+s, \omega, x) = \phi(t, \theta_s\omega, \phi(s, \omega, x)),$$

where ω a sample path in Ω of the Brownian motion $B(t)$ with invariant Wiener measure $\mathbb{P}_W$.



The one-point Markov process

- The one-point process $(x_t)_{t\geq 0}$ is associated with a family of probabilities $(\mathbb{P}_x)_{x\in X}$ with $\mathbb{P}_x(x_0 = x) = 1$ and transition probabilities

$$\hat{P}_t(x, A) = \mathbb{P}_x(x_t \in A), \ \ t \geq 0.$$

- The Fokker-Planck equation describes time-evolution of associate probability densities $p(x, t)$

$$L_t^* p := \frac{\partial p}{\partial t}(x, t) = -\frac{\partial}{\partial x}\left(f(x)p(x, t)\right) + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}(x, t).$$

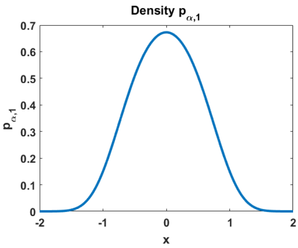
- p is called a **stationary** density if $L_t^* p = 0$.

Pitchfork Bifurcation with Additive Noise

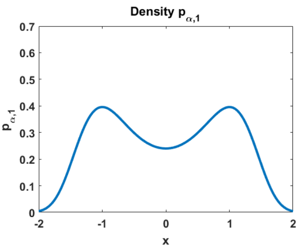
For $f_\alpha(x) = -\partial_x V_\alpha(x)$ and $\sigma > 0$:

Analytical solution for stationary density

$$p(x) = N_{\alpha,\sigma} \exp(-V_\alpha(x)/\sigma^2)$$



: $\alpha = -1$



: $\alpha = 1$

Ergodic theory

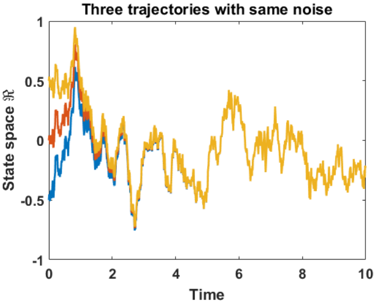
It turns out that $\rho \times \mathbb{P}_W$ is invariant (and ergodic) for the skew-product motion of the RDS with one-sided time (noise defined on $\mathbb{R}^+$). Henceforth, by **Birkhoff's Ergodic Theorem** this implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(\phi(t,\omega,x))dt \;\; = \;\; \int_X g(y)d\rho(y),$$

for almost all (x,ω).

NB: While we observe a change of the "shape" of the stationary density when α passes through zero, this is not a particularly useful/informative criterion for bifurcation. (L. Arnold branded this a phenomenological (P) bifurcation.

Trajectory point of view: synchronisation



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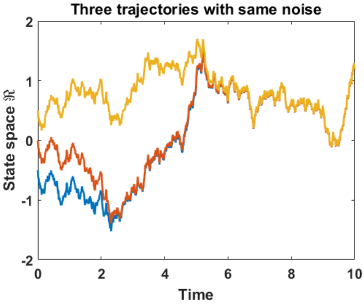


: $\alpha = 1$

Lyapunov exponent

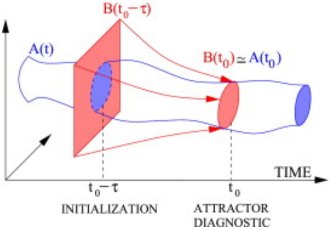
- The Lyapunov exponent of the RDS $\phi(t,\omega,x)$ is

$$\begin{aligned}\lambda &= \lim_{t\to\infty}\frac{1}{t}\log|D_x\phi(t,\omega)(x)| \\ &= N_{\alpha,\sigma}\int_{\mathbb{R}}(\alpha-3x^2)\exp\left(\frac{1}{\sigma^2}(\alpha x^2-\frac{1}{2}x^4)\right)dx \\ &= -\frac{2N_{\alpha,\sigma}}{\sigma^2}\int_{\mathbb{R}}(\alpha x-x^3)^2\exp\left(\frac{1}{\sigma^2}(\alpha x^2-\frac{1}{2}x^4)\right)dx \\ &< 0.\end{aligned}$$

- In general, Lyapunov exponents for 1D SDEs are always ≤ 0 (and rarely $= 0$).

Pullback dynamics

In the absence of any sensible convergence of behaviour in the limit where time goes to infinity (due to the assumed intrinsic randomness of the driving), in non-autonomous dynamical systems, the alternative concept of pullback-dynamics has been developed where one considers the asymptotic behaviour of $\phi(t, \theta_{-t}(\omega), x)$ as $t \to \infty$.



In order to use this concept, we need to consider two-sided time.

Random pullback attractors

A random compact set $A : \Omega \to \mathcal{K}(X)$ is called a *random pullback attractor* for the RDS (θ, φ) if

1. $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t \geq 0$ and a.a. $\omega \in \Omega$,
2. for every compact $B \subset X$, we have $\mathbb{P}$-a.s.

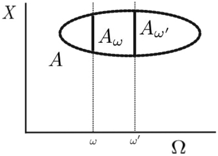
$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}\omega)B, A(\omega)) = 0. \quad (1)$$

Pullback attractors are also (weak) forward attractors: moving target for the forward dynamics.

Stationary measure versus invariant (Markov) measure

Let ρ be the stationary measure of an RDS and μ the associated invariant measure, defined as

$$\mu(A) = \int \mu_\omega(A_\omega) d\mathbb{P}(\omega),$$



where $A_\omega := \{x \in X \mid (x, \omega) \in A\}$ and

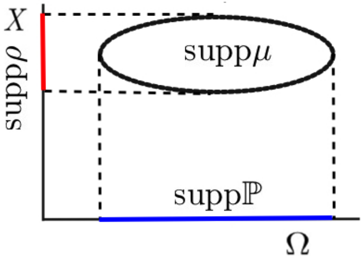
$$\mu_\omega = \lim_{t \to \infty} \phi(t, \theta_{-t}\omega)^* \rho.$$

μ is called a Markov measure as μ_ω is measurable with respect to the past (only).

Markov measure versus stationary measure

The stationary measure ρ associated to an invariant Markov measure μ is the marginal $\rho = \mu_X$, i.e. for measurable $U \subset X$

$$\rho(U) := \mu(U \times \Omega) = \int_\Omega \mu_\omega(U) d\mathbb{P}(\omega).$$



Moreover, we have $\mathbb{P}(C) = \mu(C \times X)$.

Attracting random fixed point

A consequence of negative Lyapunov exponent:

Theorem (Crauel and Flandoli 98)

For all $\alpha \in \mathbb{R}$ *and* $\sigma \in \mathbb{R} \setminus \{0\}$, *the pullback attractor of the RDS* ϕ *generated by*

$$dx = (\alpha x - x^3)\, dt + \sigma dW_t$$

is a singleton set $\{a(\omega)\}$ *and*

$$\delta_{a(\omega)} = \lim_{n \to \infty} \phi(t, \theta_{-t}\omega)^* \rho$$

$\mathbb{P}_W$*-almost surely.*

$\Rightarrow$ synchronisation: $d(\varphi(t,\omega)x_i, \varphi(t,\omega)x_j) \to 0$ as $t \to \infty$ almost surely.

Does additive noise destroy the pitchfork bifurcation?

This was the conclusion of Crauel and Flandoli 98 as for all α, there is

- Strictly negative Lyapunov exponent.
- Unique attracting random fixed point.

But does this justify their conclusion?
perhaps not... we have

$|\phi(t,\omega)x - a_\alpha(\theta_t\omega))| \leq K(\omega)\exp(\lambda t)|x - a_\alpha(\omega)|$, with $\lambda < 0$.

Uniform attractivity: $K(\omega) < \hat{K} < \infty$ iff $\alpha < 0$.

At $\alpha = 0$, the Dichotomy Spectrum crosses zero (Callaway, Doan, Lamb, Rasmussen (2017)).

Lyapunov spectrum

- Linear RDS in $\mathbb{R}^N$:
  $\phi(t,\omega)(ax_1 + bx_2) = a\phi(t,\omega)x_1 + b\phi(t,\omega)x_2$.
  Denoted as $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^{N\times N}$.
- Osceledets: (under mild assumptions) $\exists k$ Lyapunov exponents $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ and $\mathbb{R}^N = W_1(\omega) \oplus \ldots W_k(\omega)$ so that $\lambda_i := \lim_{t\to\pm\infty} \frac{1}{|t|} \ln ||\Phi(t,\omega)||$ for $0 \neq x \in W_i(\omega)$.
- But we have just seen that "bifurcation" is not necessarily associated with a change of stability in the Lyapunov spectrum.
- We claim that a better concept for this purpose is the Dichotomy spectrum

Dichotomy spectrum

- Definition: (θ, Φ) has an exponential dichotomy wrt growth rate $\gamma \in \mathbb{R}$ if there exists a splitting $\mathbb{R}^N = S(\omega) \oplus U(\omega)$, measurable and invariant ($\Phi(t,\omega)S(\omega) = S(\theta_t\omega)$, etc), satisfying for some $K, \varepsilon > 0$
  $||\Phi(t,\omega)x|| \leq Ke^{(\gamma-\varepsilon)t}||x||$, for all $t \geq 0$ n $x \in S(\omega)$.
  $||\Phi(t,\omega)x|| \geq K^{-1}e^{(\gamma+\varepsilon)t}||x||$, for all $t \geq 0$, $x \in U(\omega)$.
- Dichotomy spectrum $\Sigma := \mathbb{R} \setminus \bigcup_{\text{growth rates } \gamma}\{\gamma\}$.
- **Spectral Theorem:** $\Sigma = I_1 \cup \ldots \cup I_k$ with $I_i = \{W_i(\omega)\}_{\omega\in\Omega}$ and corresponding decomposition $\mathbb{R}^N = W_1(\omega) \oplus \ldots \cup W_k(\omega)$.
- In the pitchfork example, $\Sigma = (-\infty, \alpha]$, so that the random pitchfork bifurcation corresponds to a loss of hyperbolicity of the Dichotomy spectrum.

Finite-time Lyapunov exponents.

- $\lambda(T,\omega,x) := \frac{1}{T}\ln|D_x\phi(T,\omega)(x)|$. (random variable!)
- Lyapunov exponent is $\lambda := \lim_{T\to\infty}\lambda(T,\omega,x)$.

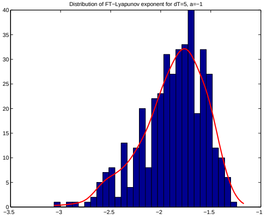
Theorem (Callaway *et al.* 2017)

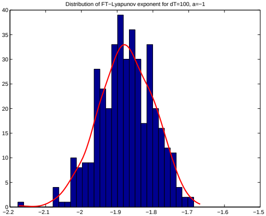
(i) If $\alpha < 0$, the random attractor is finite-time attractive:
$\lambda(T,\omega,x) \leq \alpha < 0$.
(ii) If $\alpha > 0$, the random attractor is not finite-time attractive and
$\mathbb{P}\{\omega \in \Omega : \lambda(T,\omega,x) > 0\} > 0$.

Corollary: The (negative) sign of the Lyapunov exponent can be observed *almost surely* in finite time, iff $\alpha < 0$.

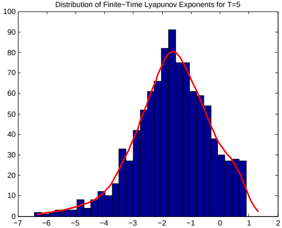
Finite-time Lyapunov spectrum

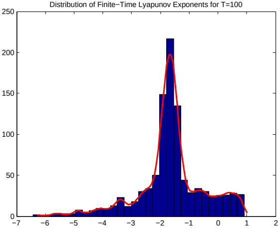
$\alpha = -1$





$\alpha = 1$





Dichotomy spectrum and finite-time Lyapunov exponents.

Theorem

Let (θ, Φ) *be a linear random dynamical system on* $\mathbb{R}^d$ *with dichotomy spectrum* Σ*, and finite-time Lyapunov exponents* $\lambda(T, \omega, x) := \frac{1}{T} \ln |D_x \phi(T, \omega)(x)|$. *Then, provided that* $\sup \Sigma < \infty$,

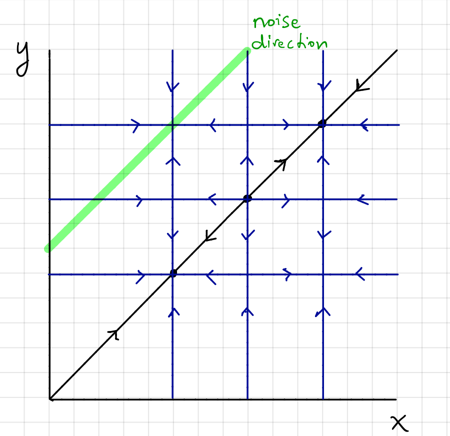
$$\lim_{T \to \infty} \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in \mathbb{R}^d \setminus \{0\}} \lambda(T, \omega, x) = \sup \Sigma$$

and, provided that $\inf \Sigma > -\infty$,

$$\text{and} \quad \lim_{T \to \infty} \operatorname*{ess\,inf}_{\omega \in \Omega} \inf_{x \in \mathbb{R}^d \setminus \{0\}} \lambda(T, \omega, x) = \inf \Sigma.$$

Dynamical view on non-uniform synchronisation: two-point motion

Consider $(x, y) \to (\phi(t, \omega, x), \phi(t, \omega, y))$.



Topological versus uniform topological equivalence.

- RDSs $\phi_1(t,\omega)$ and $\phi_2(t,\omega)$ are topologically conjugate iff $\exists$ homeomorphism $h : \Omega \times \mathbb{R} \to \mathbb{R}$ so that for all $\omega \in \Omega$, $\phi_2(t,\omega)h(\omega,x) = h(\theta_t\omega, \phi_1(t,\omega)x)$ for all t, x.
- **Theorem:** For the pitchfork example all ϕ_α are topologically equivalent.
- **Theorem:** A topological conjugacy h from ϕ_α to $\phi_{\alpha'}$ with $\mathrm{sgn}(\alpha) = -\mathrm{sgn}(\alpha')$ cannot be uniformly continuous.
  Proof: uniformly continuous conjugacies preserve local uniform attractivity.

Hopf Normal Form with Additive Noise

(Doan, Engel, Lamb Rasmussen (2018))

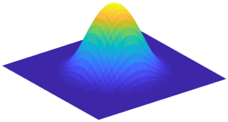
Consider the Hopf-type stochastic differential equations ,
cf also Wieczorek (2009) and Deville et al. (2011))

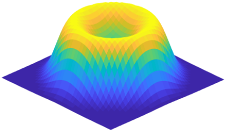
$$\begin{aligned} \mathrm{d}x &= (\alpha x - \beta y - (ax - by)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^1\,, \\ \mathrm{d}y &= (\alpha y + \beta x - (bx + ay)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^2\,, \end{aligned} \tag{2}$$

where $\nu, a, b, \sigma > 0$. The parameter b is known as the shear. This SDE has a unique stationary density

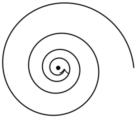
$$p(x, y) = K \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right),$$

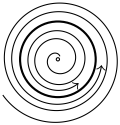
where K is a normalization constant. In the absence of noise ($\sigma = 0$) all solutions (except the origin) are attracted to a limit cycle with radius $\sqrt{2\alpha/a}$.





: $\alpha < 0$: $\alpha > 0$

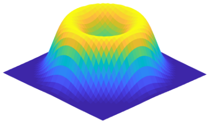




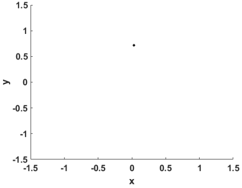
: $\alpha < 0$: $\alpha > 0$

Figure: Shape of the stationary density of (2) with noise (independent of b!) and corresponding phase portraits of the deterministic limit.

Shear-induced chaos

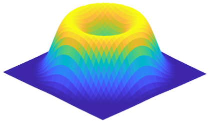


: Stationary density

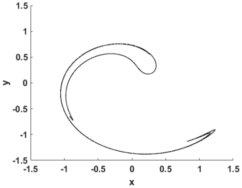


: $\alpha = -1$, $b = 1$, $T = 50$

Synchronisation



: Stationary density



: $\alpha = -1$, $b = 20$, $T = 50$

Chaos: sensitive dependence on initial conditions.

Different regimes of stability for Hopf bifurcation (2) - partially numerical

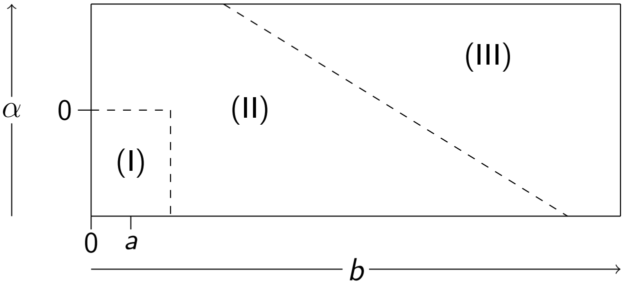
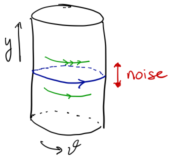


Figure: For a, β, σ fixed, we partition the (b, α)-parameter space associated with (2) schematically into three parts with different stability behaviour. Region (I) represents uniform synchronisation, region (II) non-uniform synchronisation and region (III) the absence of synchronisation and "chaos", cf (Sato, Doan, Lamb, Rasmussen (2018))

Proof of shear induced chaos in simpler setting.

(Engel, Lamb, Rasmussen (2019))

Stochastic flow on the cylinder near attracting limit cycle



$$
\begin{aligned}
dy_t &= -\alpha y_t dt + \sigma f(\vartheta_t) dW_t \\
\vartheta_t &= (1 + b y_t) dt\,,
\end{aligned}
$$

has positive Lyapunov exponent for sufficiently large b, for appropriate choises of f. This answers an open problem posed by Lin-Young (2008).

No rigorous proof yet of shear-induced chaos in the previous "Hopf" setting.

Take home messages:

- Statistical properties of the one-point motion provide only limited information about random dynamics.
- Simplest random attractor is a uniformly attractive random fixed point, but is relatively seen in SDE context.
- There are many open problems concerning the dynamics near a non-uniformly attracting random fixed point.
- It is hard to prove the existence of positive Lyapunov exponents (chaos).
- The transition from random fixed point to random chaotic attractor is poorly understood.

References

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- Mark Callaway, Thai Son Doan, Jeroen S. W. Lamb, and Martin Rasmussen, The dichotomy spectrum for random dynamical systems and pitchfork bifurcations with additive noise, *Annales de l'Institut Henri Poincaré Probabilités et Statistiques* **53**, 4 (2017), 1548–1574.