Basin Entropy: A new Method to Measure Unpredictability in Physical Systems

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In Physics we have laws that determine the time evolution of a given physical system, depending on its parameters and its initial conditions.

In multi-stable systems with many basins of attraction possessing fractal or even Wada boundaries the prediction becomes harder depending on the initial conditions.

Chaotic systems typically present fractal basins.

A small uncertainty in the initial conditions gives rise to a certain unpredictability of the final state.

The new notion of Basin Entropy provides a new quantitative way to measure the unpredictability of the final states in basins of attraction.
Quantum and Classical Uncertainty

Of course we must emphasize that classical physics is also indeterminate, in a sense.

For already in classical mechanics there was indeterminability from a practical point of view.
Sensitivity to initial conditions...
One of the sources of uncertainty

Another source of uncertainty: fractal basins
Analogy: a river basin

For want of a nail

For want of a nail the shoe was lost.
For want of a shoe the horse was lost.
For want of a horse the rider was lost.
For want of a rider the battle was lost.
For want of a battle the kingdom was lost.
And all for the want of a horseshoe nail.
A basin: if a drop falls in the region, it will go to the river
A basin of attraction is the set of initial conditions whose trajectories go to a specific attractor.
Fractality implies Unpredictability and Uncertainty
A fundamental question

Which basin is more unpredictable?

\[ H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3 \]
Unpredictability and fractal boundaries: uncertainty dimension

Smooth Boundary

\[ f \sim \varepsilon \quad \alpha = 1 \]

Fractal Boundary

\[ f \sim \varepsilon^\alpha \quad \alpha < 1 \]

\[ \alpha = D - d \]

A basin $A$ is riddled by $B$, if for every point of $A$ it is possible to find arbitrarily close points of $B$.

$\alpha = 0 \rightarrow$ Randomness of a deterministic system
Problem 1: the uncertainty dimension does not take into account how many attractors you have.
Problem 2: the uncertainty dimension does not take into account the portion of the phase space occupied by the boundary.

Both pictures have the same uncertainty dimension ($\alpha=1$).
Problems and limitations 3: Riddled basins

Problem 3: the uncertainty dimension does not distinguish among riddled basins.

\[ \alpha \approx 0 \quad \text{...but they have different structure!!} \]
The Wada property: three or more basins have a common boundary.

Wada basin boundaries: it implies more unpredictability.

\[ \ddot{x} + \delta \dot{x} + \sin x = F \cos \omega t \]
The Nusse-Yorke condition

Wada basin boundaries and basin cells

Helena E. Nusse$^{a,b}$, James A. Yorke$^{a,c}$


Palis’s Lambda Lemma
Testing for Basins of Wada

Alvar Daza¹, Alexandre Wagemakers¹, Miguel A.F. Sanjuán¹ & James A. Yorke²

$W_3 = 1 \rightarrow \text{Full Wada}$
Wada boundaries remain unaltered when the basins are merged.
Can we say that Wada basins are more unpredictable?

Vague affirmations: can we measure that unpredictability?

It seems to be a problem arising in many scientific fields...
Escape dynamics and fractal basins boundaries in the three-dimensional Earth-Moon system

Euaggelos E. Zotos
Visualizing Basins of Attraction for Different Minimization Algorithms

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ABSTRACT: We report a study of the basins of attraction for potential energy minima defined by different minimization algorithms for an atomic system. We find that whereas some minimization algorithms produce compact basins, others produce basins with complex boundaries or basins consisting of disconnected parts. Such basins deviate from the “correct” basin of attraction defined by steepest-descent pathways, and the differences can be controlled to some extent by adjustment of the maximum step size. The choice of the most convenient minimization algorithm depends on the problem in hand. We show that while L-BFGS is the fastest minimizer, the FIRE algorithm is also quite fast and can lead to less fragmented basins of attraction.
Fundamental Unpredictability in Multispecies Competition

Jef Huisman* and Franz J. Weissing‡
Exploring Classically Chaotic Potentials with a Matter Wave Quantum Probe

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• Which basin is “more fractal”?

• Which basin has a larger uncertainty?

• How can we measure the uncertainty of a basin?
Basin entropy definition


Entropy for a box: $S_i = -\sum_{j=1}^{N_A} p_{ij} \log p_{ij}$

Total entropy for $N$ boxes: $S = \sum_{i=1}^{N} S_i$

Basin entropy definition: $S_b = \frac{S}{N}$

Quantification of unpredictability: $S_b \in [0, \log N_A]$

probability(color) = proportion(color) inside the box
The three ingredients of the Basin Entropy

\[ S_b = \sum_{k=1}^{k_{\text{max}}} \frac{n_k}{\tilde{n}} \varepsilon^{\alpha_k} \log(m_k). \]

- **Boundary Size** \[ \frac{n_k}{\tilde{n}} \]
- **Uncertainty exponent** \[ \alpha_k \]
- **Number of attractors (colors)** \[ m_k \]

\( k \in [1, k_{\text{max}}] \) is the label for the different boundaries.

All these terms depend on the dynamics and are independent of the scaling box size.
Dependence with the size of the boundary

\[ \ddot{x} + \delta \dot{x} - x + x^3 = 0 \]

Duffing oscillator

Smooth boundaries (\(\alpha=1\)) but different basin entropy

\[
\log(S_b) = \alpha \log(\varepsilon) + \log \left( \log \left( \frac{N_A}{\bar{n}} \right) \right).
\]
Dependence with the uncertainty exponent

\[ H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3 \]

Hénon-Heiles System

Uncertainty exponent

\[ \log(S_b) = \alpha \log(\varepsilon) + \log \left( \log(N_A) \frac{n}{\tilde{n}} \right) \]

- Same proportion for each color 1/3, (same basin stability) but different basin entropy
- Useful also for conservative systems (escape basins).
Dependence with the number of attractors

Newton fractal:

\[ z_{n+1} = z_n - \frac{z^r - 1}{rz^{r-1}} \]

Number of attractors

The more attractors the more unpredictable (in general)
Crossing beam model for cold atoms scattering


\[ H = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \alpha_1 e^{-\beta_1 y^2} - \alpha_2 e^{-\beta_2 (x \sin \theta + y \cos \theta)^2} \]
The basin entropy indicates that for low speed the unpredictability is higher.
Escape Basins and Basin Entropy versus Energy in the conservative Hénon-Heiles Hamiltonian

\[ H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3 \]

\[ E \in [0.1665, 0.45] \]

**Nonhyperbolic regime**
- Fluctuations in \( S_b \)
- Presence of KAM islands

**Hyperbolic regime**
- Monotonous decrease in \( S_b \)
- Absence of KAM islands
Basin Entropy vs $\beta$

in the relativistic Hénon-Heiles system

Global relativistic effects in chaotic scattering

Juan D. Bernal, Jesús M. Seoane, and Miguel A. F. Sanjuán

\[ \dot{\gamma} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}} \]

\[
\begin{align*}
\dot{x} &= \frac{p}{\gamma}, \\
\dot{y} &= \frac{q}{\gamma}, \\
\dot{p} &= -x - 2xy, \\
\dot{q} &= -y - x^2 + y^2,
\end{align*}
\]
The Event Horizon Telescope collaboration made the first image of a black hole (BH shadow) at the center of the M87 galaxy (April 10, 2019).

- We can simulate BH shadows.
- **Ray-tracing**: trace photons away from camera’s lens, backwards in time.
- **Shadow** is region of image where rays are traced back to a BH.
- Rays that escape to ∞ are bright regions of image.
The Majundar-Papapetrou Binary Black Hole

General Relativity leads to Nonlinear Dynamics
Shadows of the MP binary BH

FIG. 5. Shadows cast by the static MP binary BH for different values of the separation $d$. The photons which escape to spatial infinity are plotted in green; the shadow cast by the upper (lower) BH is plotted in blue (red). These three open sets can be viewed as exit basins, defined on the image plane of a distant observer.
We have developed new methods for testing Wada basins: The Grid Method, the Merging Method and the Saddle-straddle method.

The basin entropy quantifies the final state unpredictability of dynamical systems. It constitutes a new tool for the exploration of the uncertainty in nonlinear dynamics.

We have applied these methods to different domains in Physics, such as cold atoms, shadows of binary black holes, and classical and relativistic chaotic scattering in astrophysics.

We believe that the concept of Basin Entropy will become an important tool in complex systems studies with applications in multiple scientific fields especially those with multistability and other scientific areas as well.


