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Quantum localization of chaotic systems

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CONTENTS

The Main Assertion of Stationary Quantum Chaos
(Casati, Valz-Gries, Guarneri 1980; Bohigas, Giannoni, Schmit 1984; Percival 1973)

(A1) If the system is classically integrable: **Poissonian spectral statistics**

(A2) If classically fully chaotic (ergodic): **Random Matrix Theory (RMT)** applies

- If there is an antiunitary symmetry, we have GOE statistics
- If there is no antiunitary symmetry, we have GUE statistics

(A3) If of the mixed type, in the deep semiclassical limit: we have no spectral correlations: the spectrum is a **statistically independent superposition of regular and chaotic level sequences**: The gap probability factorizes:

$$E(S) = E_1(\mu_1 S)E_2(\mu_2 S)$$

μ_j= relative fraction of phase space volume = relative density of corresponding quantum levels: μ_1 is the Poissonian, μ_2 chaotic, and $\mu_1 + \mu_2 = 1$.

According to our theory, for a two-component system, $j = 1, 2$, we have (Berry and Robnik 1984):

$$E(S) = E_1(\mu_1 S) E_2(\mu_2 S)$$

Poisson (regular) component: $E_1(S) = e^{-S}$

Chaotic (irregular) component: $E_2(S) = \mathrm{erfc}\left(\frac{\sqrt{\pi} S}{2}\right)$ (Wigner = 2D GOE)

$E(S) = E_1(\mu_1 S) E_2(\mu_2 S) = e^{-\mu_1 S} \mathrm{erfc}(\frac{\sqrt{\pi}\mu_2 S}{2})$, where $\mu_1 + \mu_2 = 1$.

Then $P(S) =$ level spacing distribution $= \frac{d^2 E(S)}{dS^2}$ and we obtain:

$P_{BR}(S) = e^{-\mu_1 S} \left(\exp(-\frac{\pi \mu_2^2 S^2}{4})(2\mu_1\mu_2 + \frac{\pi \mu_2^3 S}{2}) + \mu_1^2 \mathrm{erfc}(\frac{\mu_2 \sqrt{\pi} S}{2}) \right)$
(Berry and Robnik 1984)

This is a one parameter family of distribution functions with normalized total probability $< 1 >= 1$ and mean level spacing $< S >= 1$, whilst the second moment can be expressed in the closed form and is a function of μ_1.

THE IMPORTANT SEMICLASSICAL CONDITION

The semiclassical condition for the random matrix theory to apply in the chaotic eigenstates is that **the Heisenberg time t_H is larger than the classical transport time t_T of the system**!

The Heisenberg time of any quantum system $= t_H = \frac{2\pi\hbar}{\Delta E} = 2\pi\hbar\rho(E)$

$\Delta E = 1/\rho(E)$ is the mean energy level spacing, $\rho(E)$ is the mean level density

The quantum evolution follows the classical evolution including the chaotic diffusion up to the Heisenberg time, at longer times the destructive interference sets in and causes

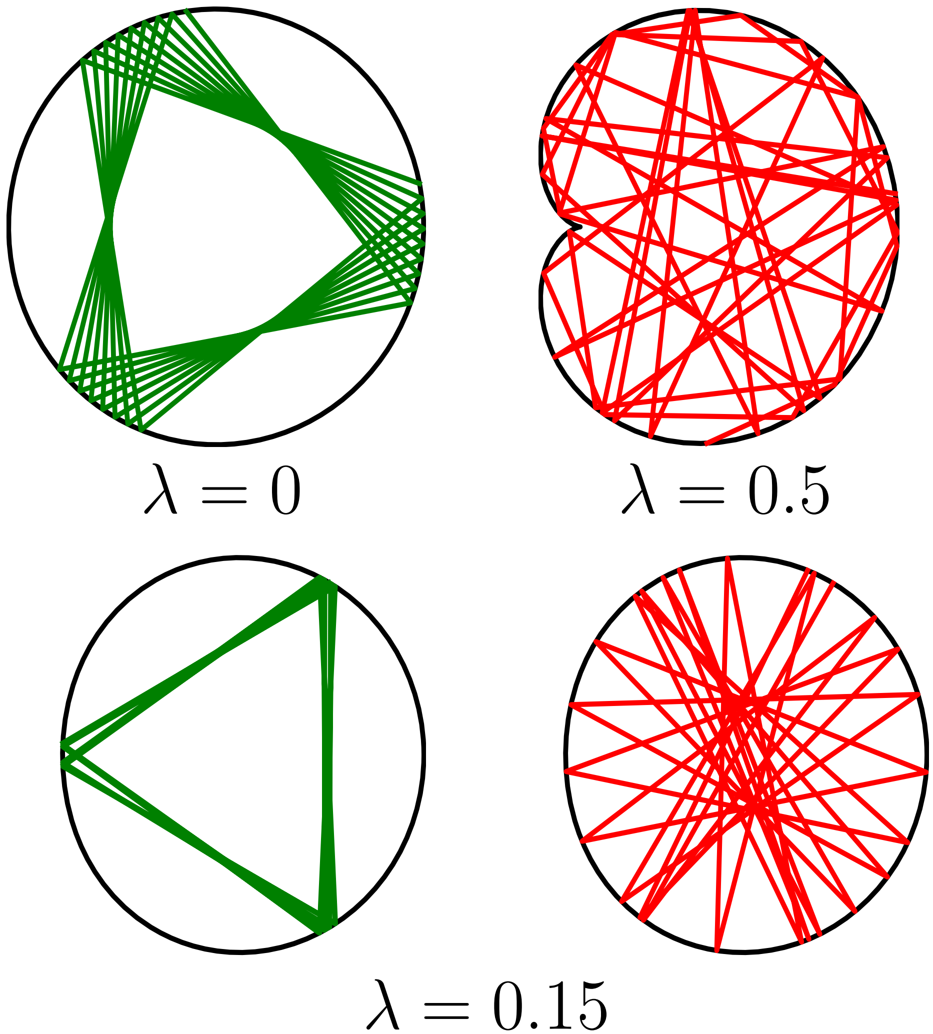
the quantum or **dynamical localization**

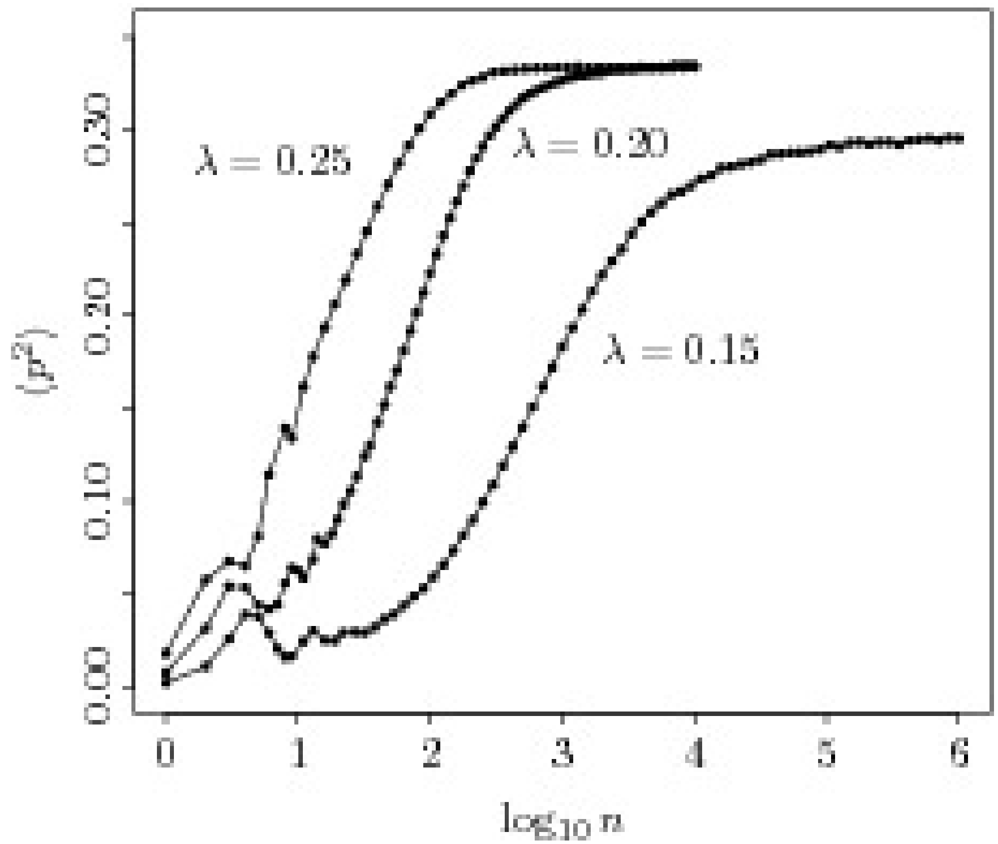
Note: $\rho(E) \propto \frac{1}{(2\pi\hbar)^N} \to \infty$ when $\hbar \to 0$, and therefore eventually $t_H \gg t_T$.

This observation applies to time-dependent and to time-independent systems.

An example of a Hamiltonian mixed-type system

One parameter family of mixed-type billiards: $w = z + \lambda z^2$, $|z| = 1$.





We show the second moment $\langle p^2 \rangle$ averaged over an ensemble of 10^6 initial conditions uniformly distributed in the chaotic component on the interval $s \in [0, \mathcal{L}/2]$ and $p = 0.$. We see that the saturation value of $\langle p^2 \rangle$ is reached at about $N_T = 10^5$ collisions for $\lambda = 0.15$, $N_T = 10^3$ collisions for $\lambda = 0.20$ and $N_T = 10^2$ for $\lambda = 0.25$. For $\lambda = 0.15$, according to the criterion at $k = 2000$ and $k = 4000$, we are still in the regime where the dynamical localization is expected. On the other hand, for $\lambda = 0.20, 0.25$ we expect extended states already at $k < 2000$.

Principle of Uniform Semiclassical Condensation (PUSC) of Wigner functions of eigenstates (Percival 1973, Berry 1977, Shnirelman 1979, Voros 1979, R. 1987-1998, Veble, R. and Liu 1999)

We study the structure of eigenstates in "quantum phase space": **The Wigner functions of eigenstates** (they are real valued but **not positive definite**):

Definition: $W_n(\mathbf{q},\mathbf{p}) = \frac{1}{(2\pi\hbar)^N}\int d^N\mathbf{X}\exp\left(-\frac{i}{\hbar}\mathbf{p}.\mathbf{X}\right)\psi_n(\mathbf{q}-\frac{\mathbf{X}}{2})\psi_n^*(\mathbf{q}+\frac{\mathbf{X}}{2})$

$(P1) \quad \int W_n(\mathbf{q},\mathbf{p})d^N\mathbf{p} = |\psi_n(\mathbf{q})|^2$

$(P2) \quad \int W_n(\mathbf{q},\mathbf{p})d^N\mathbf{q} = |\phi_n(\mathbf{p})|^2$

$(P3) \quad \int W_n(\mathbf{q},\mathbf{p})d^N\mathbf{q}\, d^N\mathbf{p} = 1$

$(P4) \quad (2\pi\hbar)^N\int d^N\mathbf{q}\, d^N\mathbf{p} W_n(\mathbf{q},\mathbf{p})W_m(\mathbf{q},\mathbf{p}) = \delta_{nm}$

$(P5) \quad |W_n(\mathbf{q},\mathbf{p})| \le \frac{1}{(\pi\hbar)^N}$ (Baker 1958)

$(P6 = P4) \quad \int W_n^2(\mathbf{q},\mathbf{p})d^N\mathbf{q}\, d^N\mathbf{p} = \frac{1}{(2\pi\hbar)^N}$

$(P7) \quad \hbar \to 0: \quad W_n(\mathbf{q},\mathbf{p}) \to (2\pi\hbar)^N W_n^2(\mathbf{q},\mathbf{p}) > 0$

Quantum or dynamical localization of chaotic eigenstates

If we are not sufficiently deep in the semiclassical regime of sufficiently small effective Planck constant $\hbar_{eff}$, which e.g. in billiards means not at sufficiently high energies, we observe

dynamical localization of chaotic eigenstates,

of the Wigner functions in the phase space $(\mathbf{q}, \mathbf{p})$, which is the cause for the deviation from the RMT statistics: *The reason: Heisenberg time is shorter than the classical transport time.*

The control parameter for this transition from localized to extended chaotic eigenstates is

$$\alpha = \frac{t_H}{t_T} = \frac{\text{Heisenberg time}}{\text{classical transport time}}$$

Dynamical localization: $\alpha \leq 1$

Extended eigenstates: $\alpha \geq 1$

Dynamically localized chaotic states are semiempirically well described by the Brody level spacing distribution: (Izrailev 1988,1989, Prosen and R. 1993/4, Batistić and R. 2010-2013)

$$P_B(S) = C_1 S^\beta \exp\left(-C_2 S^{\beta+1}\right), \quad F_B(S) = 1 - W_B(S) = \exp\left(-C_2 S^{\beta+1}\right),$$

where $\beta \in [0,1]$ and the two parameters C_1 and C_2 are determined by the two normalizations $< 1 >=< S >= 1$, and are given by

$C_1 = (\beta+1)C_2, \quad C_2 = \left(\Gamma\left(\frac{\beta+2}{\beta+1}\right)\right)^{\beta+1}$ with $\Gamma(x)$ being the Gamma function. If we have extended chaotic states $\beta = 1$ and RMT applies, whilst in the strongly localized regime $\beta = 0$ and we have Poissonian statistics. The corresponding gap probability is

$$E_B(S) = \frac{1}{(\beta+1)\Gamma\left(\frac{\beta+2}{\beta+1}\right)} Q\left(\frac{1}{\beta+1}, \left(\Gamma\left(\frac{\beta+2}{\beta+1}\right) S\right)^{\beta+1}\right)$$

$Q(\alpha, x)$ is the incomplete Gamma function: $Q(\alpha,x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$.

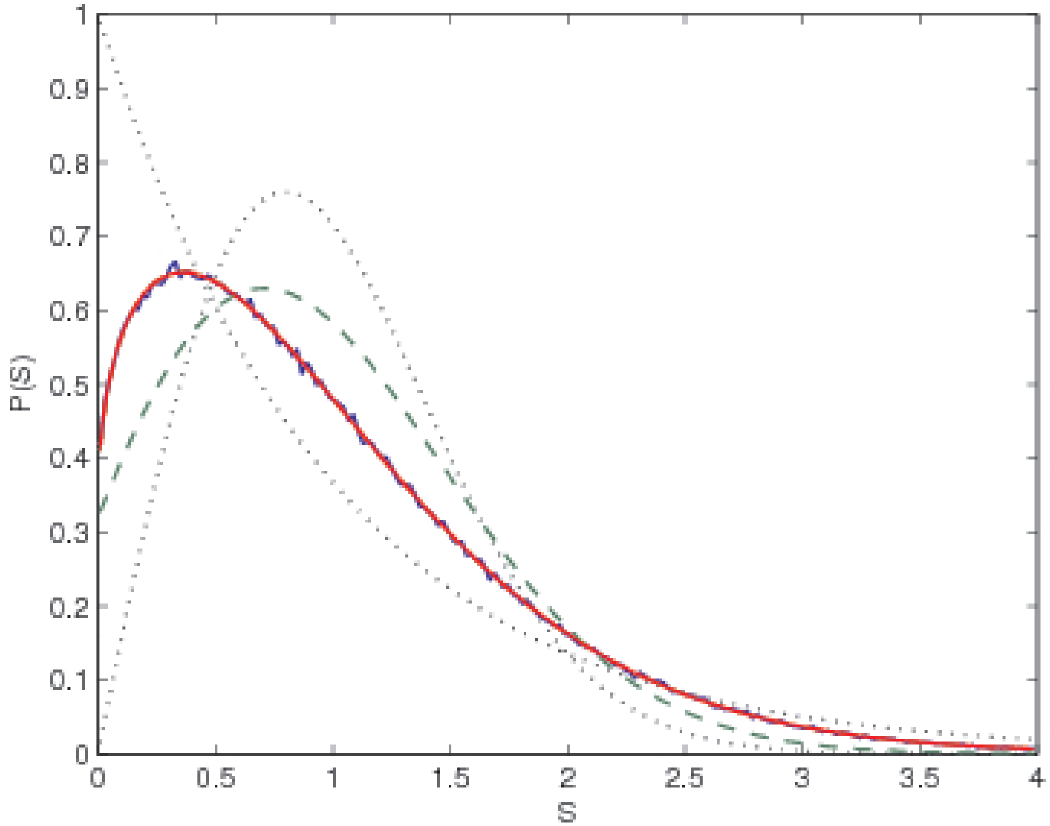
Back to the mixed-type systems: The BRB theory: BR-Brody
(Prosen and Robnik 1993/1994, Batistić and Robnik 2010)

We have divided phase space $\mu_1 + \mu_2 = 1$ and localization β:

$$E(S) = E_r(\mu_1 S) E_c(\mu_2 S) = \exp(-\mu_1 S) E_{Brody}(\mu_2 S)$$

and the level spacing distribution $P(S)$ is:

$$P(S) = \frac{d^2 E_r}{dS^2} E_c + 2 \frac{dE_r}{dS} \frac{dE_c}{dS} + E_r \frac{d^2 E_c}{dS^2}$$



The level spacing distribution for the billiard $\lambda = 0.15$, compared with the analytical formula for BRB (red full line) with parameter values $\rho_1 = 0.183$, $\beta = 0.465$ and $\sigma = 0$. The dashed red curve close to the full red line is BRB with classical $\rho_1 = 0.175$ is not visible, as it overlaps completely with the quantum case $\rho_1 = 0.183$. The dashed curve far away from the red full line is just the BR curve with the classical $\rho_1 = 0.175$. The Poisson and GOE curves (dotted) are shown for comparison. The agreement of the numerical spectra with BRB is perfect. In the histogram we have 650000 objects, and the statistical significance is extremely large.

Separating the regular and chaotic eigenstates in a mixed-type billiard system

The idea:

Introduce the quantum phase space analogous to the classical billiard phase space in Poincaré-Birkhoff coordinates $(s, p = \sin\alpha)$, by using the Husimi functions in the same space.

Look at the overlap of the quantum eigenstates with the classical regular and classically chaotic component(s), and thus separate the regular and chaotic eigenstates and also the corresponding energy eigenvalues.

Then perform the spectral statistical analysis separately for the regular and chaotic level sequences.

We find: Poisson for regular and Brody for chaotic eigenstates.

The model billiard: $w = z + \lambda z^2$

$$\Delta\psi + k^2\psi = 0, \quad \psi|_{\partial\mathcal{B}} = 0. \tag{1}$$

$$u(s) = \mathbf{n}\cdot\nabla_{\mathbf{r}}\psi\left(\mathbf{r}(s)\right), \tag{2}$$

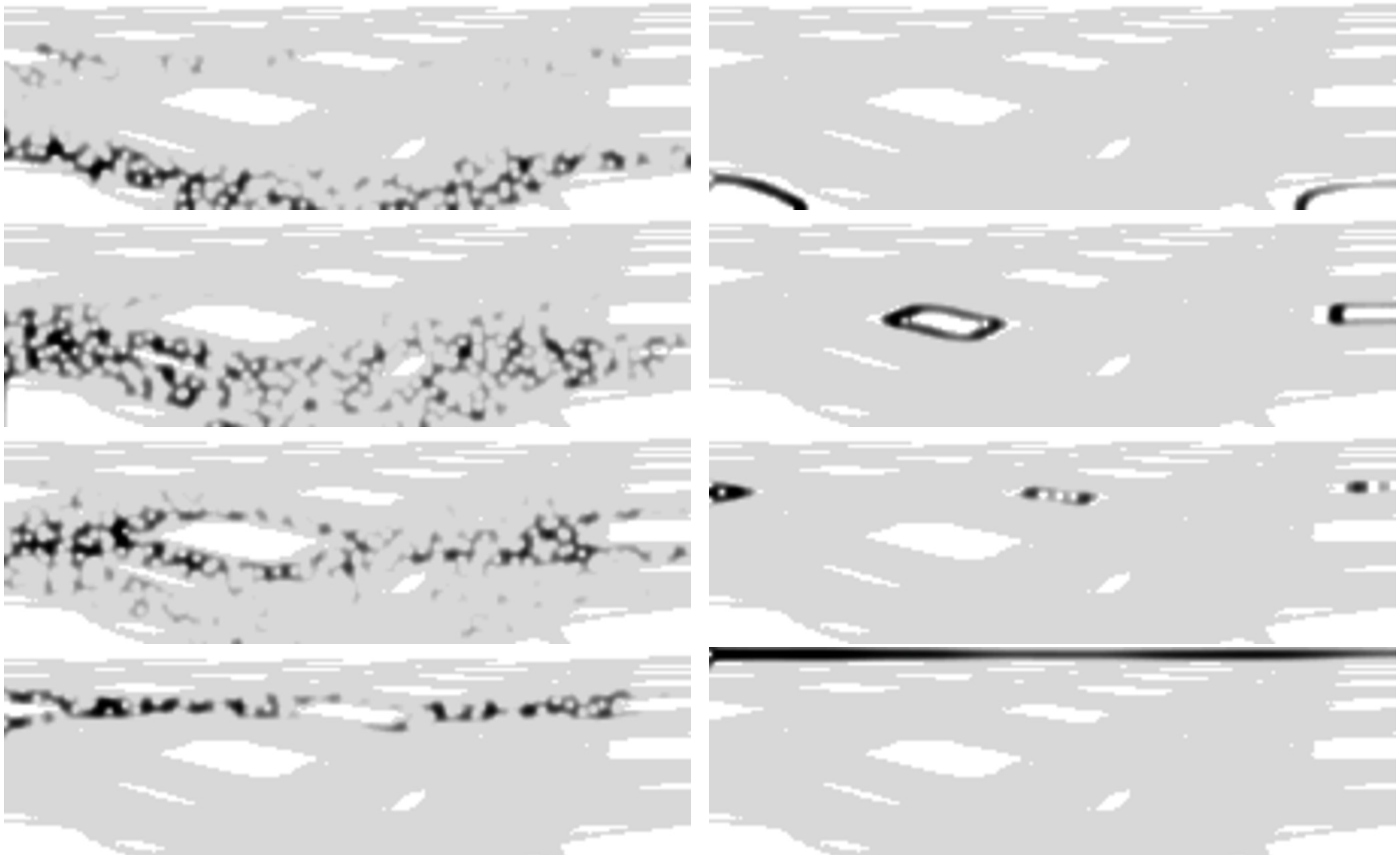
$$u(s) = -2\oint dt\; u(t)\;\mathbf{n}\cdot\nabla_{\mathbf{r}}G(\mathbf{r},\mathbf{r}(t)). \tag{3}$$

$$G(\mathbf{r},\mathbf{r}') = -\frac{i}{4}H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|), \tag{4}$$

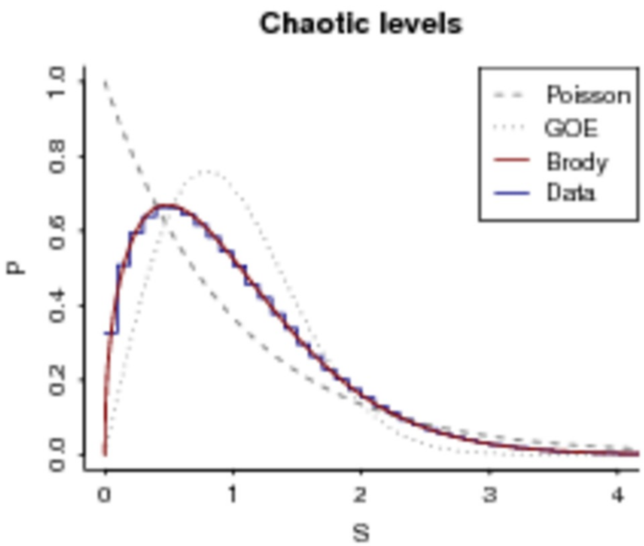
$$\psi_j(\mathbf{r}) = -\oint dt\; u_j(t)\; G\left(\mathbf{r},\mathbf{r}(t)\right). \tag{5}$$

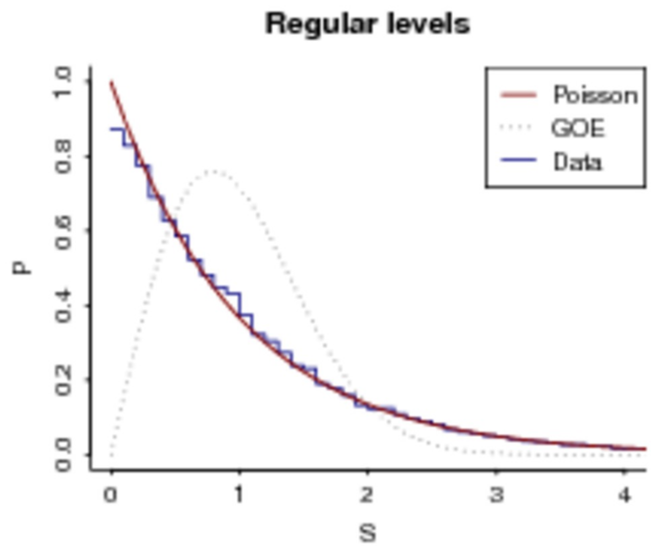
$$c_{(q,p),k}(s) = \sum_{m\in\mathbf{Z}} \exp\{i\,k\,p\,(s-q+m\mathcal{L})\}\exp\left(-\frac{k}{2}(s-q+m\mathcal{L})^2\right). \tag{6}$$

$$H_j(q,p) = \left|\int_{\partial\mathcal{B}} c_{(q,p),k_j}(s)\; u_j(s)\; ds\right|^2, \quad M = \sum_{i,j} H_{i,j}\; A_{i,j}. \tag{7}$$



Examples of chaotic (left) and regular (right) states in the Poincaré-Husimi representation. k_j (M) from top down are: chaotic: k_j $(M) =$ 2000.0021815 (0.978), 2000.0181794 (0.981), 2000.0000068 (0.989), 2000.0258600 (0.965); regular: k_j $(M) =$ 2000.0081402 (-0.987), 2000.0777155 (-0.821), 2000.0786759 (-0.528), 2000.0112417 (-0.829). The gray background is the classically chaotic invariant component. We show only one quarter of the surface of section $(s, p) \in [0, \mathcal{L}/2] \times [0, 1]$, because due to the reflection symmetry and time-reversal symmetry the four quadrants are equivalent. $\lambda = 0.15$.





Separation of levels of the billiard $\lambda = 0.15$ using the classical criterion $M_t = 0.431$. (a; left) The level spacing distribution for the chaotic subspectrum after unfolding, in perfect agreement with the Brody distribution $\beta = 0.444$. (b; right) The level spacing distribution for the regular part of the spectrum, after unfolding, in excellent agreement with Poisson.

The localization measures of chaotic eigenstates:
recent work by Batistić and R.2013, Batistić, Lozej and R. 2018,2019,2020

A: localization measure based on **the information entropy** of the Husimi quasi-probability distribution:

Calculate normalized Husimi distribution $H(q,p)$ on the phase space (q,p) and then the information entropy for each chaotic eigenstate

$$I = -\int dq\, dp\, H(q,p) \ln\left((2\pi\hbar)^N H(q,p)\right)$$

and **define:** $A = \frac{\exp I}{\Omega_C/(2\pi\hbar)^N}$ **(= entropy localization measure)**

where Ω_C = phase space volume on which $H(q,p)$ is defined, and the averaging is over a large number of consecutive chaotic eigenstates.

- Uniform distribution $H = 1/\Omega_C$: $A = 1$ (extendedness)

- Strongest localization in a single Planck cell: $H = 1/(2\pi\hbar)^N$

$$I = \ln\left((2\pi\hbar)^N H\right) = 0 \text{ and } A = (2\pi\hbar)^N/\Omega_C = 1/N_{Ch}(E) \approx 0$$

C: localization measure based on **the correlations** of the Husimi quasi-probability distribution:

Calculate normalized Husimi distribution $H_m(q,p)$ for each chaotic eigenstate labeled by m, and then the correlation matrix for large number of consecutive chaotic eigenstates:

$$C_{nm} = \frac{1}{Q_n Q_m} \int dq\, dp\, H_n(q,p)\, H_m(q,p)$$

where $Q_n = \sqrt{\int dq\, dp\, H_n^2(q,p)}$ is the normalizing factor

and define

$$C = \langle C_{nm} \rangle \qquad (= \textbf{correlation localization measure})$$

where the averaging is over a large number of consecutive chaotic eigenstates

nIPR: localization measure in terms of **the normalized inverse participation ratio** of the normalized Husimi quasi-probability distribution:

$$nIPR = \frac{1}{N}\frac{1}{\sum_{i,j} H_{ij}^2}$$

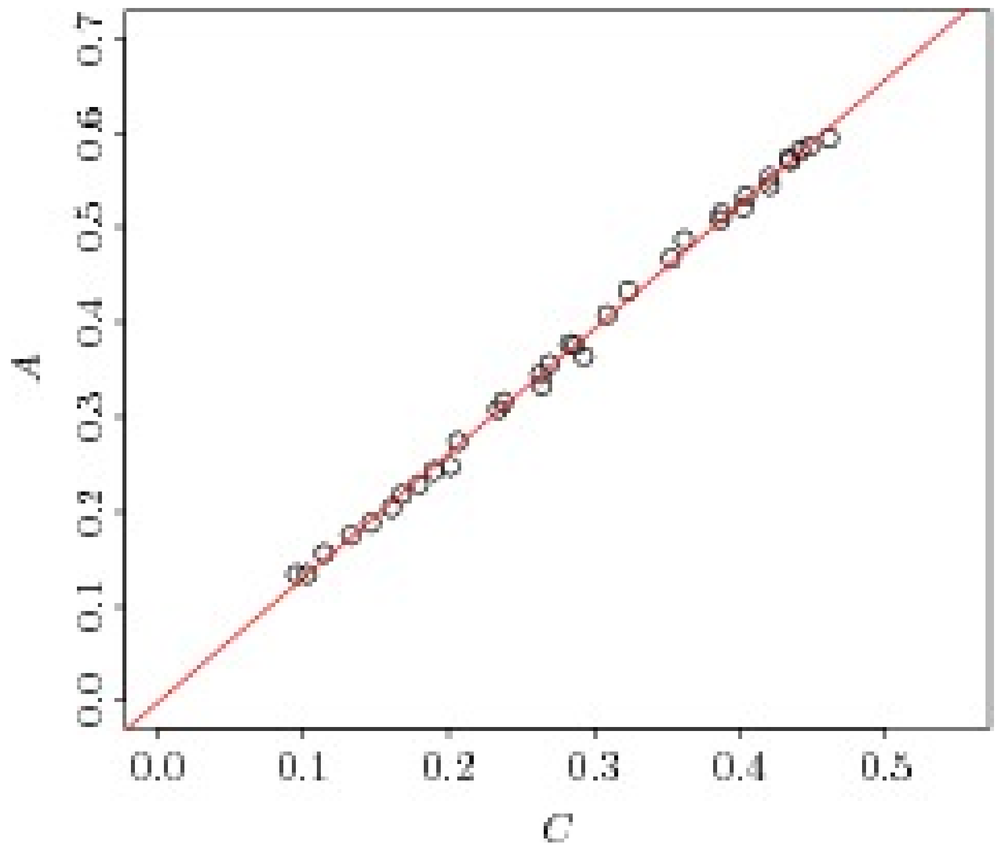
$N =$ the number of cells (i,j)

Normalization: $\sum_{i,j} H_{ij} = 1$

In case of uniformly extended states: $H_{ij} = \frac{1}{N}$, and thus $nIPR = 1$.

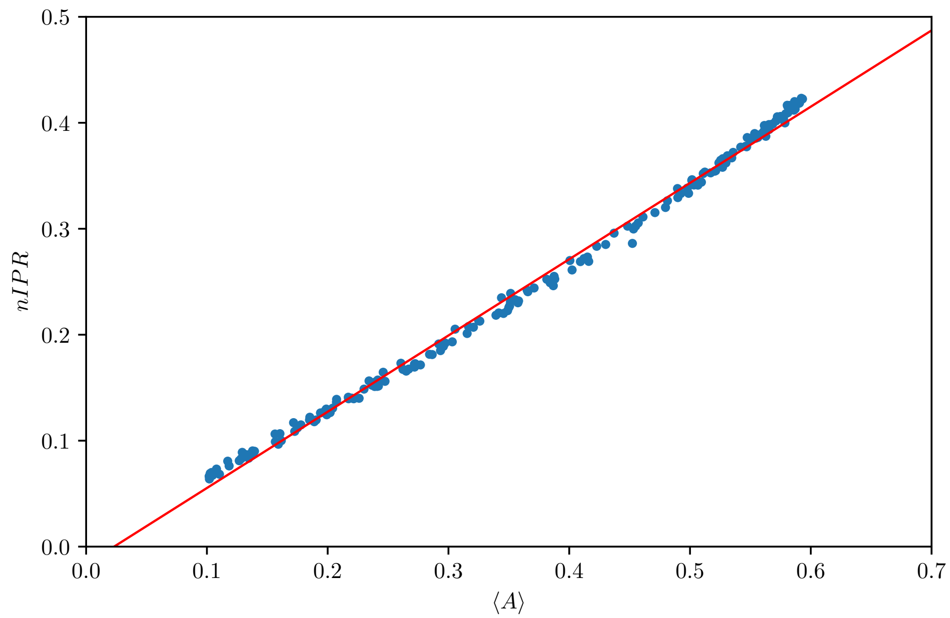
In case of maximally localized states: $H_{ij} = 1$ in only one cell, $nIPR = 1/N \approx 0$.

Surprisingly and satisfactory: The two localization measures A and C are linearly related and thus equivalent !



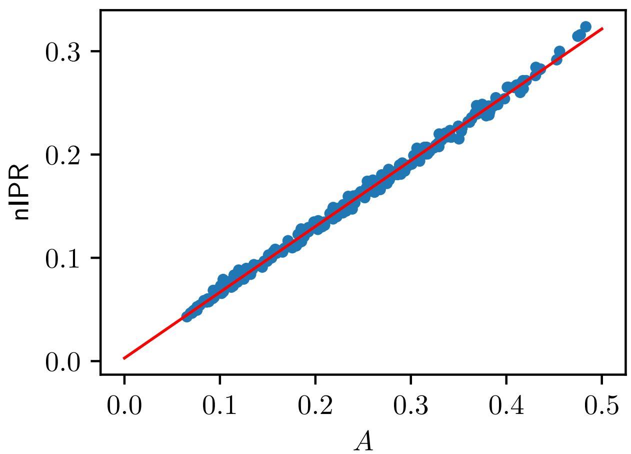
Linear relation between the two entirely different localization measures, namely the entropy measure A and the correlation measure C, calculated for **the mixed-type billiard** $\lambda = 0.15$ and $k \approx 2000$ and $k \approx 4000$.

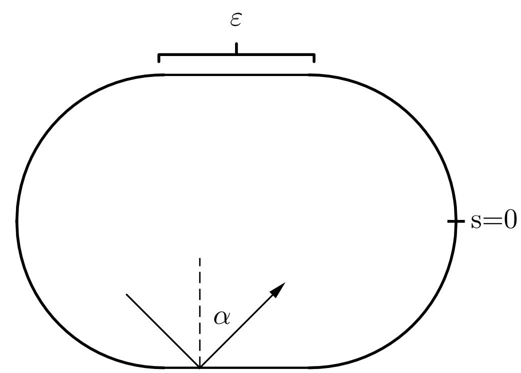
Surprisingly and satisfactory: The two localization measures A and nIPR are linearly related and thus equivalent !



Linear relation between the two entirely different localization measures, namely the entropy measure A and the normalized inverse participation ratio $nIPR$ of the Husimi functions, calculated for several different **mixed-type billiards at various** λ **and** k.

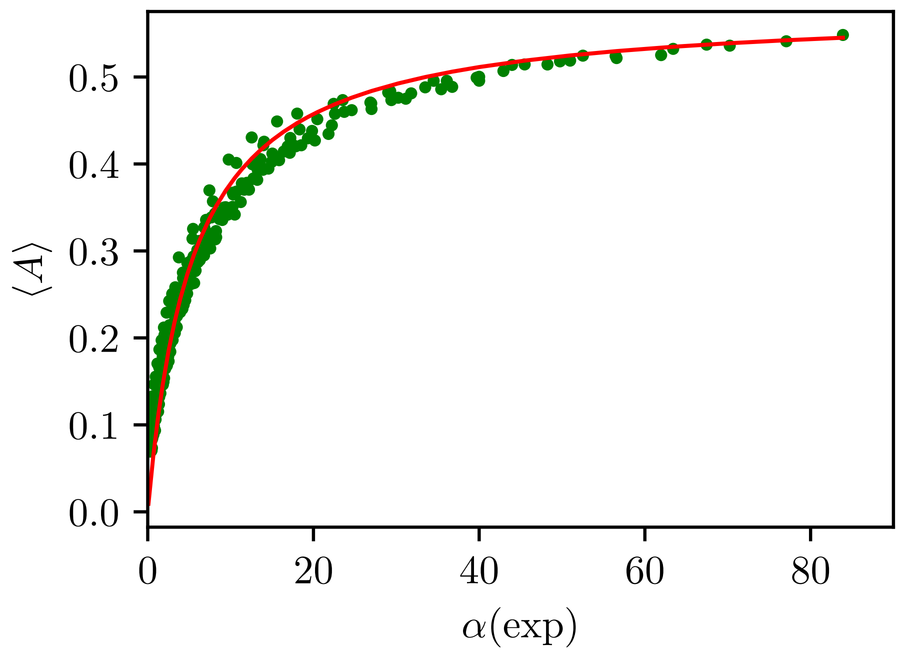
Surprisingly and satisfactory: The two localization measures A and nIPR are linearly related and thus equivalent !





Linear relation between the two entirely different localization measures, namely the entropy measure A and the normalized inverse participation ratio $nIPR$ of the Husimi functions, calculated for **several different stadium billiards at various** ϵ **and** k.

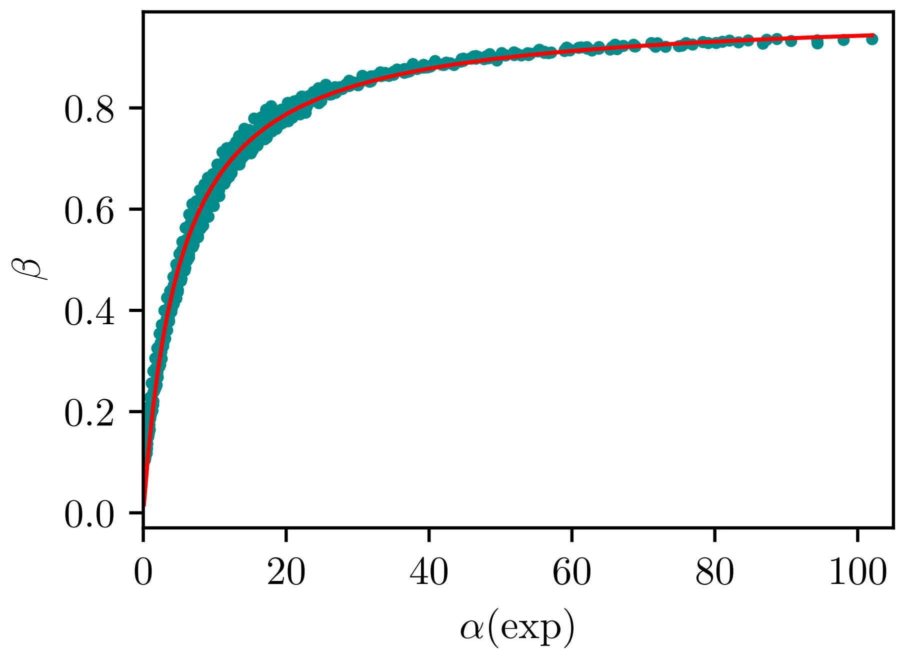
The localization measure $\langle A \rangle$ of the Poincaré-Husimi functions for several different stadium billiards at various ϵ and k as a function of $\alpha = t_H/t_T$



$$\langle A \rangle = A_\infty \frac{s\alpha}{1+s\alpha},$$

where the values of the two parameters are $A_\infty = 0.58$ and $s = 0.19$.

Stadium billiard: Brody parameter β is a unique function of $\alpha = t_H/t_T = \frac{2k}{N_T}$

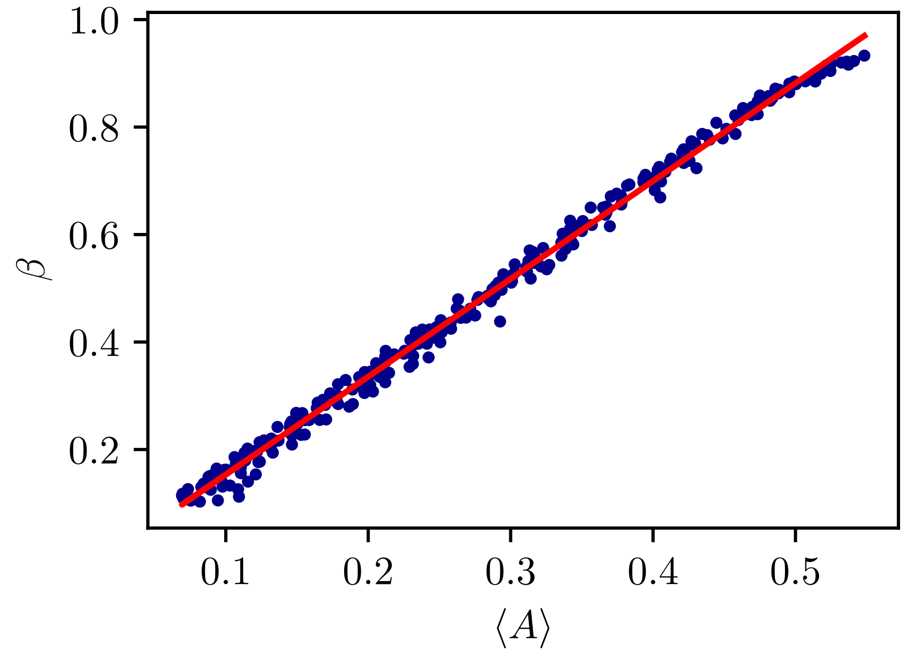


The level repulsion exponent β as a function of the entropy localization measure $\langle A \rangle$ for variety of stadia of different shapes ϵ and energies $E = k^2$:

$$\beta = \beta_\infty \frac{s\alpha}{1+s\alpha}.$$

the coefficient s depends on the definition of N_T and α. For the exponential law we found $\beta_\infty = 0.98$ and $s = 0.20$. **See: F. Borgonovi, G. Casati and B. Li, PRL 77 (1996) 4744**

Stadium billiard: Brody parameter β is a unique linear function of $\langle A \rangle$



The level repulsion exponent β as a function of the entropy localization measure $\langle A \rangle$ for variety of stadia of different shapes ϵ and energies $E = k^2$:

The localization measure A has beta distribution $P(A)$:

$$P(A) = CA^a(A_0 - A)^b,$$

where A_0 is the upper limit of the interval $[0, A_0]$ on which $P(A)$ is defined, and $A_0 \approx 0.7$.

The two exponents a and b are positive real numbers, while C is the normalization constant such that $\int_0^{A_0} P(A)\, dA = 1$, i.e.

$$C^{-1} = A_0^{a+b+1} B(a+1, b+1)$$

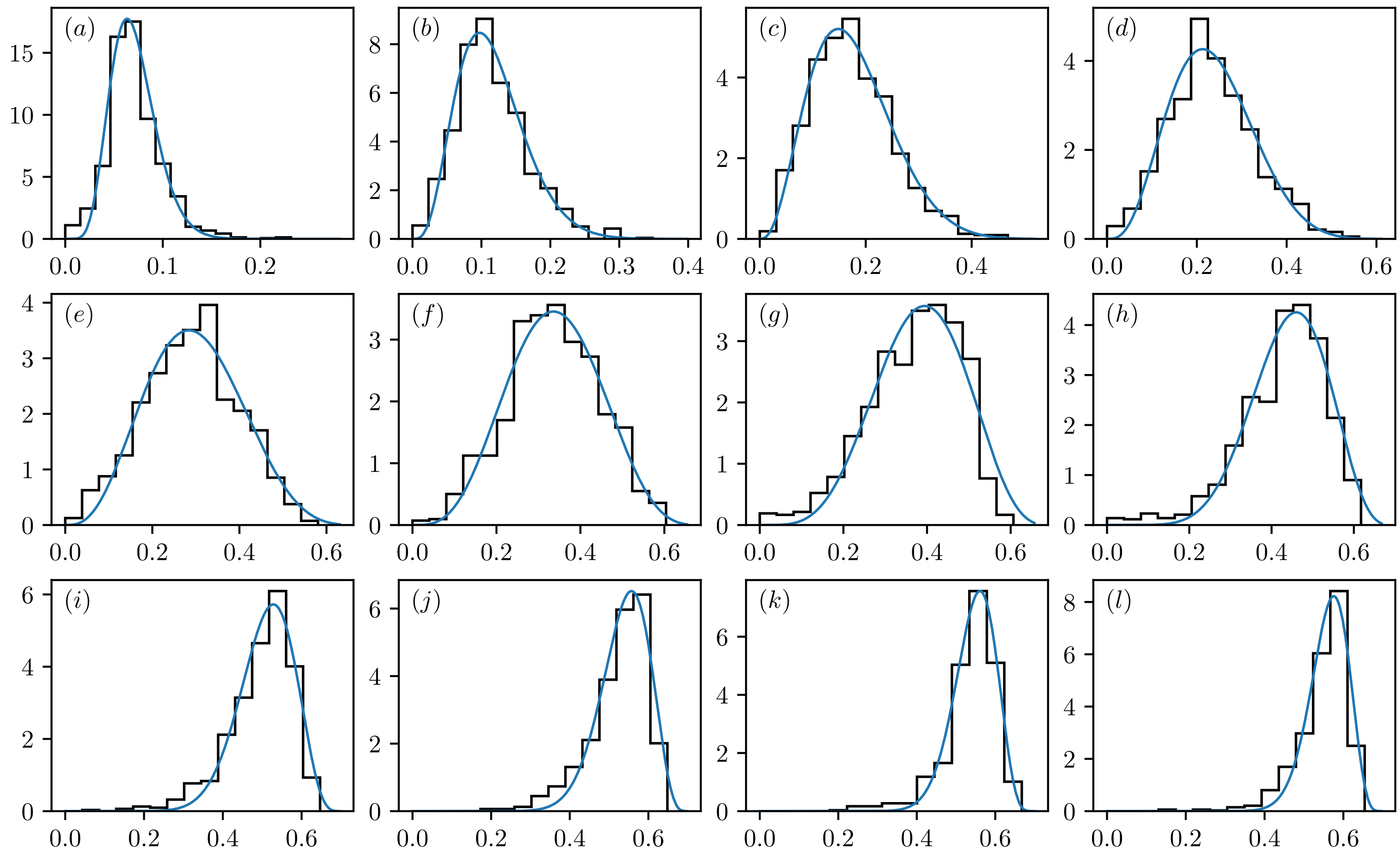
where $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ is the beta function.

$$\langle A \rangle = A_0 \tfrac{a+1}{a+b+3}, \text{ and } \langle A^2 \rangle = A_0^2 \tfrac{(a+2)(a+1)}{(a+b+4)(a+b+3)}$$

and therefore for the standard deviation σ

$$\sigma^2 = A_0^2 \tfrac{(a+2)(b+2)}{(a+b+4)(a+b+3)^2}.$$

such that asymptotically $\sigma \approx A_0 \frac{\sqrt{b+2}}{a}$ when $a \to \infty$.



Stadium billiard: The distributions $P(A)$ of the entropy localization measure A for $k_0 = 3440$ and various ϵ (from (a) to (l)): 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.1, 0.14, 0.16, 0.18, 0.2.

5. Discussion and conclusions

• The Principle of Uniform Semiclassical Condensation of Wigner functions of eigenstates leads to the idea that in the sufficiently deep semiclassical limit the spectrum of a mixed type system can be described as a statistically independent superposition of regular and chaotic level sequences.

• As a result of that the gap probabilities $E(S)$ factorize and the level spacing distribution and other statistics can be calculated in a closed form.

• At lower energies we see quantum or dynamical localization of chaotic eigenstates.

• The level spacing distribution of localized chaotic eigenstates is excellently described by the Brody distribution with $\beta \in [0, 1]$.

• In the mixed type systems regular and chaotic eigenstates can be separated: the regular obey Poisson, the localized chaotic states obey the Brody.

• The localization measures of the chaotic eigenstates **A**, **C** and **nIPR** are equivalent.

• **A** has the beta distribution $P(A)$ if there is no stickiness.

• The Brody level repulsion exponent β is a linear function of the localization measure $\langle A \rangle$.

• β is universal function of α, the ratio of the Heisenberg time and classical transport time.

Some recent publications

Robnik M 2020 *Recent Advances in Quantum Chaos of Generic Systems: wave chaos of mixed-type systems.* in *Encyclopedia of Complex Systems* Ed. R.A. Meyers (Berlin: Springer) ISBN 978-3-642-27737-5, ISBN 3-642-27737-3

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Batistić B, Lozej Č and Robnik M 2019 Phys. Rev. E **100** 062208

Batistić B, Lozej Č and Robnik M 2018 Nonlin. Phen. in Complex Syst. (Minsk) **21** 225

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